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STABILITY UNVIGINTIC FUNCTIONAL EQUATION IN MULTI-BANACH SPACES

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Abstract. In this paper we prove the Hyers-Ulam Stability of Unvigintic Functional Equations in Multi-Banach Spaces.

1 Introduction

The issue of stability of functional equations has appeared in connection with a question that Ulam [12] asked in 1940. Hyers [6], by using direct method, brilliantly gave a partial answer for the case of the additive Cauchy functional equation for mappings between Banach Spaces. This result was then improved by Aoki [1] and Rassias [11], who weakened the condition for the bound of the norm of Cauchy difference. The stability phenomenons proved in [6] and [11] were named Hyers-Ulam and Hyers-Ulam-Rassias stability duo to the high influence of Hyers and Rassias on this area of research.
Let us recall some basic concepts concerning Multi-Banach Spaces.
The Multi-Banach Spaces were first investigated by Dales and Polyakov [5]. Theory of Multi-Banach Spaces is similar to operator sequence space and has some connections with operator spaces and Banach Spaces. In 2007 H.G. Dales and M.S. Moslehian [2] first proved the stability of mappings and also gave some examples on multi-normed spaces. The asymptotic aspects of the quadratic functional equations in multi-normed spaces were investigated by M.S. Moslehian, K. Nikodem and D. Popa [7] in 2009. In last two decades, the stability of functional equations on multi-normed spaces was proved by many mathematicians (see, [3, 9, 10, 13, 14]).
Let $(\wp, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\wp^k$ the linear space $\wp \oplus \wp \oplus \wp \oplus ... \oplus \wp$ consisting of $k-$ tuples $(x_1, ..., x_k)$ where $x_1, ..., x_k \in \wp$.

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The linear operations on $\wp^k$ are defined coordinate wise. The zero element of either $\wp$ or $\wp^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, ..., k\}$ and by Ψ_k the group of permutations on k symbols.

Definition 1.1. [2] A multi-norm on $\left\{\wp^k : k \in \mathbb{N}\right\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\wp^k$ for each $k \in \mathbb{N}, \|x\|_1 = \|x\|$ for each $x \in \wp$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(1) $\left\|\left(x_{\sigma(1)}, ..., x_{\sigma(k)}\right)\right\|_k = \|(x_1...x_k)\|_k$, for $\sigma \in \Psi_k, x_1, ..., x_k \in \wp$;
(2) $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|) \|(x_1...x_k)\|_k$
for $\alpha_1...\alpha_k \in \mathbb{C}, x_1, ..., x_k \in \wp$;
(3) $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$, for $x_1, ..., x_{k-1} \in \wp$;
(4) $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$ for $x_1, ..., x_{k-1} \in \wp$.

In this case, we say that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space.
Suppose that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space, and take $k \in \mathbb{N}$. We need the following two properties of multi - norms. They can be found in [2].

$$(a)\ \|(x, ...x)\|_k = \|x\| , \forall x \in \wp,$$

$$(b) \max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, ..., x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\| , \forall x_1, ..., x_k \in \wp.$$

It follows from (b) that if $(\wp, \|.\|)$ is a Banach Space, then $(\wp^k, \|.\|_k)$ is a Banach Space for each $k \in \mathbb{N}$;
In this case, $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-banach space.

The following fixed point theorem proved by Diaz and Margolis [4] plays an important role in proving our theorem:

Theorem 1.2. [4] *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;
(ii) The sequence $\{\mathcal{J}^n x\}$ is convergent to a fixed point y^* of $\mathcal{J}$;
(iii) y^* is the unique fixed point of T in the set $Y = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;
(iv) $d(y, y^*) \leq \frac{1}{1-\mathcal{L}} d(y, \mathcal{J} y)$ for all $y \in Y$.

In this paper, we prove the Hyers-Ulam Stability of Unvigintic Functional Equation
$f(x + 11y) - 21f(x + 10y) + 210f(x + 9y) - 1330f(x + 8y) + 5985f(x + 7y)$
$-20349f(x + 6y) + 54264f(x + 5y) - 116280f(x + 4y) + 203490f(x + 3y)$

$$-293930f(x+2y)+352716f(x+y)-352716f(x)+293930f(x-y)$$
$$-203490f(x-2y)+116280f(x-3y)-54264f(x-4y)+20349f(x-5y)$$
$$-5985f(x-6y)+1330f(x-7y)-210f(x-8y)+21f(x-9y)-f(x-10y)=21!f(y), \tag{1.1}$$

where $21! = 51090942170000000000$ in Multi-Banach Space.

2 Hyers-Ulam Stability of Unvigintic Functional Equation (1.1) in Multi-Banach Spaces

Theorem 2.1. *Let $\mathcal{A}$ be an linear space and let $\left((\mathcal{B}^k, \|.\|_k) : K \in \mathbb{N}\right)$ be a multi-Banach space. Suppose that η is a non-negative real number and $f : \mathcal{A} \to \mathcal{B}$ be a function fulfills*

$$\sup_{k\in\mathbb{N}} \|(\mathcal{G}f(x_1,y_1),...,\mathcal{G}f(x_k,y_k))\|_k \leq \eta \tag{2.1}$$

$x_1,...,x_k,y_1,...,y_k \in \mathcal{A}$. *Then there exists a unique Unvigintic function $\mathcal{V}_{21} : \mathcal{A} \to \mathcal{B}$ such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1)-\mathcal{V}_{21}(x_1),...,f(x_k)-\mathcal{V}_{21}(x_k))\|_k \leq \frac{2097153}{21!(2097151)}\eta. \tag{2.2}$$

forall $x_i \in \mathcal{A}$, where $i = 1,2,...,k$.

Proof. Replacing $x_j = 0$, $y_j = 2x_j$ and $x_j = 11x_j$, $y_j = x_j$ where $j = 1,2,..,k$ in (2.1) and subtracting the two resulting equations, we get

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(21f(21x_1)-230f(20x_1)+1330f(19x_1)-5796f(18x_1)+20349f(17x_1)\\ &-55384f(16x_1)+116280f(15x_1)-198835f(14x_1)+293930f(13x_1)\\ &-367080f(12x_1)+352716f(11x_1)-260015f(10x_1)+203490f(9x_1)\\ &-178296f(8x_1)+54264f(7x_1)-66861f(6x_1)+5985f(5x_1)-91770f(4x_1)\\ &+210f(3x_1)+(58765-21!)f(2x_1)-(1+21!)f(x_1),...,21f(21x_k)\\ &-230f(20x_k)+1330f(19x_k)-5796f(18x_k)+20349f(17x_k)\\ &-55384f(16x_k)+116280f(15x_k)-198835f(14x_k)+293930f(13x_k)\\ &-367080f(12x_k)+352716f(11x_k)-260015f(10x_k)+203490f(9x_k)\\ &-178296f(8x_k)+54264f(7x_k)-66861f(6x_k)+5985f(5x_k)-91770f(4x_k)\\ &\qquad+210f(3x_k)+(58765-21!)f(2x_k)-(1+21!)f(x_k)))\|_k \leq 2\eta\end{aligned} \tag{2.3}$$

$\forall x_1,...,x_k \in \mathcal{A}$. Replacing (x_j,y_j) with $(10x_j,x_j)$ where $j = 1,2,..,k$ in (2.1), further multiplying the resulting inequality by 21, and subtracting the obtained result from (2.3), we arrive

$$\begin{aligned}&\sup_{k\in\mathbb{N}} \|(211f(20x_1)-3080f(19x_1)+22134f(18x_1)-105336f(17x_1)+371945f(16x_1)\\ &-1023264f(15x_1)+2243045f(14x_1)-3979360f(13x_1)+5805450f(12x_1)\\ &-7054320f(11x_1)+7147021f(10x_1)-5969040f(9x_1)+4094994f(8x_1)\\ &-2387616f(7x_1)+1206405f(6x_1)-421344f(5x_1)+33915f(4x_1)-27720f(3x_1)\\ &+(63175-21!)f(2x_1)+22(21!)f(x_1),..,\\ &211f(20x_k)-3080f(19x_k)+22134f(18x_k)-105336f(17x_k)+371945f(16x_k)\end{aligned}$$

$-1023264f(15x_k)+2243045f(14x_k)-3979360f(13x_k)+5805450f(12x_k)$
$-7054320f(11x_k)+7147021f(10x_k)-5969040f(9x_k)+4094994f(8x_k)$
$-2387616f(7x_k)+1206405f(6x_k)-421344f(5x_k)+33915f(4x_k)-27720f(3x_k)$

$$+(63175-21!)f(2x_k)+22(21!)f(x_k)))\|_k \leq 23\eta \tag{2.4}$$

$\forall x_1,...,x_k \in \mathcal{A}$. Replacing (x_j,y_j) with $(9x_j,x_j)$ where $j=1,2,..,k$ in (2.1), further multiplying the resulting inequality by 211, and subtracting the obtained result from (2.4), we arrive
$\sup_{k\in\mathbb{N}}\|(1351f(19x_1)-22176f(18x_1)+175294f(17x_1)-890890f(16x_1)$
$+3270375f(15x_1)-9206659f(14x_1)+20555720f(13x_1)-37130940f(12x_1)$
$+54964910f(11x_1)-67276055f(10x_1)+68454036f(9x_1)-57924236f(8x_1)$
$+40548774f(7x_1)-23328675f(6x_1)+11028360f(5x_1)-4259724f(4x_1)$
$+1235115f(3x_1)-(217455+21!)f(2x_1)+233(21!)f(x_1),..,1351f(19x_k)$
$-22176f(18x_k)+175294f(17x_k)-890890f(16x_k)+3270375f(15x_k)$
$-9206659f(14x_k)+20555720f(13x_k)-37130940f(12x_k)+54964910f(11x_k)$
$-67276055f(10x_k)+68454036f(9x_k)-57924236f(8x_k)+40548774f(7x_k)$
$-23328675f(6x_k)+11028360f(5x_k)-4259724f(4x_k)+1235115f(3x_k)$

$$-(217455+21!)f(2x_k)+233(21!)f(x_k)))\|_k \leq 234\eta \tag{2.5}$$

$\forall x_1,...,x_k \in \mathcal{A}$. Replacing (x_j,y_j) with $(8x_j,x_j)$ where $j=1,2,..,k$ in (2.1), further multiplying the resulting inequality by 1351, and subtracting the obtained result from (2.5), we arrive
$\sup_{k\in\mathbb{N}}\|(6195f(18x_1)-108416f(17x_1)+905940f(16x_1)-4815360f(15x_1)$
$+18284840f(14x_1)-52754944f(13x_1)+119963340f(12x_1)-219950080f(11x_1)$
$+329823375f(10x_1)-408065280f(9x_1)+418595080f(8x_1)-356550656f(7x_1)$
$+251586315f(6x_1)-146065920f(5x_1)+69050940f(4x_1)-26256384f(3x_1)$
$+(7866929-21!)f(2x_1)+1584(21!)f(x_1),..,6195f(18x_k)$
$-108416f(17x_k)+905940f(16x_k)-4815360f(15x_k)+18284840f(14x_k)$
$-52754944f(13x_k)+119963340f(12x_k)-219950080f(11x_k)+329823375f(10x_k)$
$-408065280f(9x_k)+418595080f(8x_k)-356550656f(7x_k)+251586315f(6x_k)$
$-146065920f(5x_k)+69050940f(4x_k)-26256384f(3x_k)$

$$+(7866929-21!)f(2x_k)+1584(21!)f(x_k)))\|_k \leq 1585\eta \tag{2.6}$$

$\forall x_1,...,x_k \in \mathcal{A}$. Replacing (x_j,y_j) with $(7x_j,x_j)$ where $j=1,2,..,k$ in (2.1), further multiplying the resulting inequality by 6195, and subtracting the obtained result from (2.6), we arrive
$\sup_{k\in\mathbb{N}}\|(21679f(17x_1)-395010f(16x_1)+3423990f(15x_1)-18792235f(14x_1)$
$+73307111f(13x_1)-216202140f(12x_1)+500404520f(11x_1)$
$-930797175f(10x_1)+1412831070f(9x_1)-1766480540f(8x_1)$
$+1828524964f(7x_1)-1569310035f(6x_1)+1114554630f(5x_1)-651303660f(4x_1)$
$+309902901f(3x_1)-(118065031+21!)f(2x_1)+7779(21!)f(x_1),..,$
$21679f(17x_k)-395010f(16x_k)+3423990f(15x_k)-18792235f(14x_k)$
$+73307111f(13x_k)-216202140f(12x_k)+500404520f(11x_k)$

$-930797175f(10x_k)+1412831070f(9x_k)-1766480540f(8x_k)$
$+1828524964f(7x_k)-1569310035f(6x_k)+1114554630f(5x_k)-651303660f(4x_k)$

$$+309902901f(3x_k)-(118065031+21!)f(2x_k)+7779(21!)f(x_k)))\|_k \leq 7780\eta \tag{2.7}$$

$\forall x_1,...,x_k \in \mathcal{A}$. Replacing (x_j,y_j) with $(6x_j,x_j)$ where $j=1,2,..,k$ in (2.1), further multiplying the resulting inequality by 21679, and subtracting the obtained result from (2.7), we arrive
$\sup_{k\in\mathbb{N}}\|(60249f(16x_1)-1128600f(15x_1)+10040835f(14x_1)-56441704f(13x_1)$
$+224943831f(12x_1)-675984736f(11x_1)+1590036945f(10x_1)$
$-2998628640f(9x_1)+4605627930f(8x_1)-5818005200f(7x_1)$
$+6077220129f(6x_1)-5257553840f(5x_1)+3760134371f(4x_1)$
$-2210475960f(3x_1)+(1053771635-21!)f(2x_1)+29458(21!)f(x_1),..,$
$60249f(16x_k)-1128600f(15x_k)+10040835f(14x_k)-56441704f(13x_k)$
$+224943831f(12x_k)-675984736f(11x_k)+1590036945f(10x_k)$
$-2998628640f(9x_k)+4605627930f(8x_k)-5818005200f(7x_k)$
$+6077220129f(6x_k)-5257553840f(5x_k)+3760134371f(4x_k)-2210475960f(3x_k)$

$$+(1053771635-21!)f(2x_k)+29458(21!)f(x_k)))\|_k \leq 29459\eta \tag{2.8}$$

$\forall x_1,...,x_k \in \mathcal{A}$. Replacing (x_j,y_j) with $(5x_j,x_j)$ where $j=1,2,..,k$ in (2.1), further multiplying the resulting inequality by 60249, and subtracting the obtained result from (2.8), we arrive
$\sup_{k\in\mathbb{N}}\|(136629f(15x_1)-2611455f(14x_1)+23689466f(13x_1)-135646434f(12x_1)$
$+550022165f(11x_1)-1679314791f(10x_1)+4007125080f(9x_1)$
$-7654441080f(8x_1)+11890983370f(7x_1)-15173566160f(6x_1)$
$+15993172200f(5x_1)-13947588970f(4x_1)+10036940760f(3x_1)$
$-(5871850915+21!)f(2x_1)+89707(21!)f(x_1),...,136629f(15x_k)$
$-2611455f(14x_k)+23689466f(13x_k)-135646434f(12x_k)$
$+550022165f(11x_k)-1679314791f(10x_k)+4007125080f(9x_k)$
$-7654441080f(8x_k)+11890983370f(7x_k)-15173566160f(6x_k)$
$+15993172200f(5x_k)-13947588970f(4x_k)+10036940760f(3x_k)$

$$-(5871850915+21!)f(2x_k)+89707(21!)f(x_k)))\|_k \leq 89708\eta \tag{2.9}$$

$\forall x_1,...,x_k \in \mathcal{A}$. Replacing (x_j,y_j) with $(4x_j,x_j)$ where $j=1,2,..,k$ in (2.1), further multiplying the resulting inequality by 136629, and subtracting the obtained result from (2.9), we arrive
$\sup_{k\in\mathbb{N}}\|(257754f(14x_1)-5002624f(13x_1)+46070136f(12x_1)$
$-267702400f(11x_1)+1100948730f(10x_1)-3406910976f(9x_1)$
$+8232779040f(8x_1)-15911651840f(7x_1)+24985659190f(6x_1)$
$-32195192960f(5x_1)+34214953300f(4x_1)-29940704640f(3x_1)$
$+(21113059740-21!)f(2x_1)+226336(21!)f(x_1),..,$
$257754f(14x_k)-5002624f(13x_k)+46070136f(12x_k)$
$-267702400f(11x_k)+1100948730f(10x_k)-3406910976f(9x_k)$

$+8232779040f(8x_k) - 15911651840f(7x_k) + 24985659190f(6x_k)$
$-32195192960f(5x_k) + 34214953300f(4x_k) - 29940704640f(3x_k)$

$$+(21113059740 - 21!)f(2x_k) + 226336(21!)f(x_k)))\|_k \leq 226337\eta \qquad (2.10)$$

$\forall x_1, ..., x_k \in \mathcal{A}$. Replacing (x_j, y_j) with $(3x_j, x_j)$ where $j = 1, 2, .., k$ in (2.1), further multiplying the resulting inequality by 257754, and subtracting the obtained result from (2.10), we arrive
$\sup_{k\in\mathbb{N}} \|(410210f(13x_1) - 8058204f(12x_1) + 75110420f(11x_1) - 441708960f(10x_1)$
$+1838125170f(9x_1) - 5753984016f(8x_1) + 14059725530f(7x_1) - 27459289440f(6x_1)$
$+43512311920f(5x_1) - 56356193740f(4x_1) + 59430597530f(3x_1)$
$-(49403537340 + 21!)f(2x_1) + 484090(21!)f(x_1), .., 410210f(13x_k) - 8058204f(12x_k)$
$+75110420f(11x_k) - 441708960f(10x_k) + 1838125170f(9x_k) - 5753984016f(8x_k)$
$+14059725530f(7x_k) - 27459289440f(6x_k) + 43512311920f(5x_k) - 56356193740f(4x_k)$

$$+59430597530f(3x_k) - (49403537340 + 21!)f(2x_k) + 484090(21!)f(x_k)))\|_k \leq 484091\eta \qquad (2.11)$$

$\forall x_1, ..., x_k \in \mathcal{A}$. Replacing (x_j, y_j) with $(2x_j, x_j)$ where $j = 1, 2, .., k$ in (2.1), further multiplying the resulting inequality by 410210, and subtracting the obtained result from (2.11), we arrive
$\sup_{k\in\mathbb{N}} \|(556206f(12x_1) - 11033680f(11x_1) + 103870340f(10x_1) - 616981680f(9x_1)$
$+2592969064f(8x_1) - 8191295500f(7x_1) + 20153785260f(6x_1) - 39415741680f(5x_1)$
$+61761724710f(4x_1) - 76909669540f(3x_1) + (73024457570 - 21!)f(2x_1)$
$+894300(21!)f(x_1), .., 556206f(12x_k) - 11033680f(11x_k) + 103870340f(10x_k)$
$-616981680f(9x_k) + 2592969064f(8x_k) - 8191295500f(7x_k) + 20153785260f(6x_k)$
$-39415741680f(5x_k) + 61761724710f(4x_k) - 76909669540f(3x_k)$

$$+(73024457570 - 21!)f(2x_k) + 894300(21!)f(x_k)))\|_k \leq 894301\eta \qquad (2.12)$$

$\forall x_1, ..., x_k \in \mathcal{A}$. Replacing (x_j, y_j) with (x_j, x_j) where $j = 1, 2, .., k$ in (2.1), further multiplying the resulting inequality by 556206, and subtracting the obtained result from (2.12), we arrive
$\sup_{k\in\mathbb{N}} \|(646646f(11x_1) - 12932920f(10x_1) + 122216094f(9x_1) - 724243520f(8x_1)$
$+3010137130f(7x_1) - 9288423144f(6x_1) + 21930999090f(5x_1) - 40102398340f(4x_1)$
$+56393997660f(3x_1) - (58482664240 + 21!)f(2x_1) + 1450506(21!)f(x_1), ...,$
$646646f(11x_k) - 12932920f(10x_k) + 122216094f(9x_k)$
$-724243520f(8x_k) + 3010137130f(7x_k) - 9288423144f(6x_k)$
$+21930999090f(5x_k) - 40102398340f(4x_k) + 56393997660f(3x_k)$

$$-(58482664240 + 21!)f(2x_k) + 1450506(21!)f(x_k)))\|_k \leq 1450507\eta \qquad (2.13)$$

$\forall x_1, ..., x_k \in \mathcal{A}$. Replacing (x_j, y_j) with $(0, x_j)$ where $j = 1, 2, .., k$ in (2.1), further multiplying the resulting inequality by 646646, and subtracting the obtained result from (2.13), we arrive

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{1}{2097152}f(2x_1)-f(x_1),...,\frac{1}{2097152}f(2x_k)-f(x_k)\right)\right\|_k \leq \frac{2097153}{21!(2097152)} \tag{2.14}$$

forall $x_1,...,x_k \in \mathcal{A}$. Let $\Psi = \{l : \mathcal{A} \to \mathcal{B} | l(0) = 0\}$ and introduce the generalized metric d defined on Ψ by

$$d(l,m) = \inf\left\{\Psi \in [0,\infty] | \sup_{k\in\mathbb{N}} \|l(x_1)-m(x_1),...,l(x_k)-m(x_k)\|_k \leq \Psi \quad \forall \quad x_1,...,x_k \in \mathcal{A}\right\}$$

Then it is easy to show that (Ψ, d) is a generalized complete metric space, See [8].
We define an operator $\mathcal{J} : \Psi \to \Psi$ by

$$\mathcal{J}l(x) = \frac{1}{2^{21}}l(2x) \qquad x \in \mathcal{A}$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $l,m \in \Psi$, let $\Psi \in [0,\infty]$ be an arbitary constant with $d(l,m) \leq \Psi$. From the definition if follows that

$$\sup_{k\in\mathbb{N}} \|l(x_1)-m(x_1),...,l(x_k)-m(x_k)\|_k \leq \Psi \quad x_1,...,x_k \in \mathcal{A}.$$

Therefore, $\sup_{k\in\mathbb{N}} \|(\mathcal{J}l(x_1)-\mathcal{J}m(x_1),...,\mathcal{J}l(x_k)-\mathcal{J}m(x_k))\|_k \leq \frac{1}{2^{21}}\Psi$ $x_1,...,x_k \in \mathcal{A}$. Hence,it holds that

$$d(\mathcal{J}l,\mathcal{J}m) \leq \frac{1}{2^{21}}\Psi d(\mathcal{J}l,\mathcal{J}m) \leq \frac{1}{2^{21}}d(l,m)$$

$\forall l,m \in \Psi$.
This Means that $\mathcal{J}$ is strictly contractive operator on Ψ with the Lipschitz constant $L = \frac{1}{2^{21}}$.
By (2.14), we have $d(\mathcal{J}f,f) \leq \frac{2097153}{21!(2097152)}\eta$. Applying the Theorem 2.2 in [**?**], we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $\mathcal{V}_{21} : \mathcal{A} \to \mathcal{B}$ such that

$$\mathcal{V}_{21}(2x) = 2^{21}\mathcal{V}_{21}(x) \qquad \forall x \in \mathcal{A}.$$

Moreover, we have $d\left(\mathcal{J}^n f, \mathcal{V}_{21}\right) \to 0$, which implies

$$\mathcal{V}_{21}(x) = \lim_{n\to\infty} \mathcal{J}^n f(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{21n}}$$

for all $x \in \mathcal{A}$.
Also, $d(f,\mathcal{V}_{21}) \leq \frac{1}{1-\mathcal{L}}d(\mathcal{J}f,f)$ implies the inequality

$$d(f,\mathcal{V}_{21}) \leq \frac{2097153}{21!(2097151)}\eta.$$

Doing $x_1 =, ..., = x_k = 2^n x$ and $y_1 =, ..., = y_k = 2^n y$ in (2.1) and dividing by 2^{21n}. Now, applying the property (a) of multi-norms, we have

$$\begin{aligned}\|\mathcal{G}\mathcal{V}_{21}(x,y)\| &= \lim_{n\to\infty} \frac{1}{2^{21n}} \|\mathcal{G}f(2^n x, 2^n y)\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^{21n}} = 0\end{aligned}$$

for all $x, y \in \mathcal{A}$. The uniqueness of $\mathcal{V}_{21}$ follows from the fact that $\mathcal{V}_{21}$ is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0,\infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{V}_{21}(x_1), ..., f(x_k) - \mathcal{V}_{21}(x_k))\|_k \leq \ell$$

for all $x_1, ..., x_k \in \mathcal{A}$.
Hence the proof. □

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